Continuous-space model of computation

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Motivations

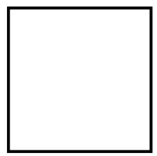
- Investigate computational power of a novel model of computation
- Relationship between models of computation and scientific theories

CSM definition

- Images are the basic data units in the CSM

- A **complex-valued image** (or simply, an image) is a complex-valued function on the real unit square

$$f : [0, 1] \times [0, 1] \mapsto \mathbb{C}$$



CSM definition

A **continuous space machine** is a quintuple $M = (D, L, I, P, O)$, where

$D = (m, n)\,,\;\; D \in \mathbb{N} \times \mathbb{N}$: grid dimensions

$L = ((s_\xi, s_\eta)\,, (a_\xi, a_\eta)\,, (b_\xi, b_\eta))$: addresses $\boldsymbol{sta}$, $\boldsymbol{a}$, and $\boldsymbol{b}$

$I = \left\{ \left(\iota_{1_\xi}, \iota_{1_\eta}\right)\,, \ldots, \left(\iota_{k_\xi}, \iota_{k_\eta}\right) \right\}$: addresses of the k input images

$P = \left\{ \left(\pi_1, p_{1_\xi}, p_{1_\eta}\right)\,, \ldots, \left(\pi_r, p_{r_\xi}, p_{r_\eta}\right) \right\}, \;\; \pi_j \in (\{$h, v, $*$, $\cdot$, $+$, ρ, st, ld, br, hlt$\} \cup \mathcal{N}) \subset \mathcal{I}$: the r programming symbols and their addresses

$O = \left\{ \left(o_{1_\xi}, o_{1_\eta}\right)\,, \ldots, \left(o_{l_\xi}, o_{l_\eta}\right) \right\}$: addresses of the l output images.

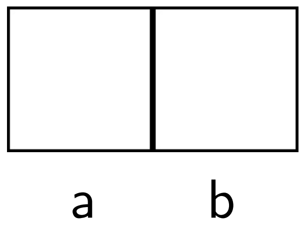
Also, $(s_\xi, s_\eta), (a_\xi, a_\eta), (b_\xi, b_\eta), (\iota_{k'_\xi}, \iota_{k'_\eta}), (p_{r'_\xi}, p_{r'_\eta}), (o_{l'_\xi}, o_{l'_\eta}) \in \{0, \ldots, m-1\} \times \{0, \ldots, n-1\}$ for all $k'_\xi, k'_\eta \in \{1, \ldots, k\}$, $r'_\xi, r'_\eta \in \{1, \ldots, r\}$, $l'_\xi, l'_\eta \in \{1, \ldots, l\}$.

A CSM **configuration** is a pair $\langle c, g \rangle$

c is an address called the control, $g = ((i_{0\,0}, 0, 0), \ldots, (i_{m-1\,n-1}, m-1, n-1))$

CSM operations

h	: horizontal 1-D Fourier transform
v	: vertical 1-D Fourier transform
$*$	: complex conjugate
$\cdot$	: multiply two images (point by point multiplication)
$+$	: add two images (complex addition)
ρ | z_l | z_u	: image filter using lower and upper amplitude threshold images z_l and z_u.



CSM operations

st	$p1$	$p2$	$p3$	$p4$

: $p1, p2, p3, p4 \in \mathbb{N}$; copy the image in **a** into the rectangle of images whose bottom left-hand corner address is ($p1$,$p3$) and whose top right-hand corner address is ($p2$, $p4$).

ld	$p1$	$p2$	$p3$	$p4$

: $p1, p2, p3, p4 \in \mathbb{N}$; copy into **a** the rectangle of images whose bottom left-hand corner address is ($p1$,$p3$) and whose top right-hand corner address is ($p2$, $p4$).

CSM operations

| br | $p1$ | $p2$ | : $p1, p2 \in \mathbb{N}$; unconditionally branch to the image at address $(p1,\ p2)$.

| hlt | : halt.

| | : move to the next grid image (ignore images that do not represent a programming symbol).

Complexity measures

Computational complexity measures are used to analyse CSM instances

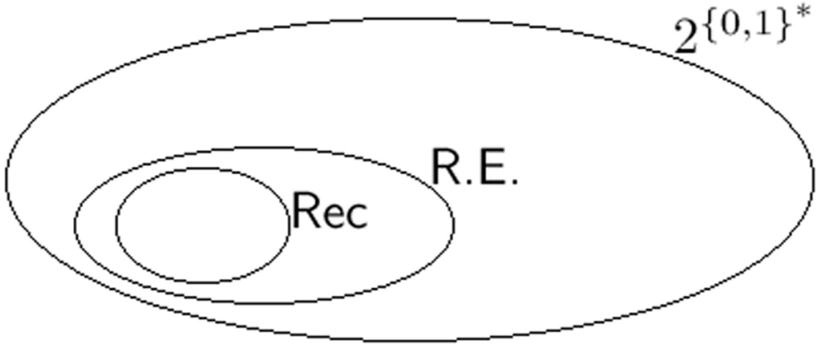
- TIME = number of computation steps
- SPACE = number of images in grid
- RESOLUTION = max spatial resolution, relative to some unit image
- RANGE = number of bits required to represent the values in the set f' where

$$f : [0, 1] \times [0, 1] \mapsto f' \subseteq \mathbb{C}$$

Symbols, words, languages

$$\{0,1\}$$
$$\{0,1\}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$$
$$L \subseteq \{0,1\}^*$$

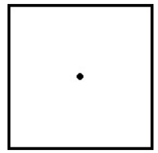
Given L and $w \in \{0,1\}^*$, is $w \in L$?



Representing data as images

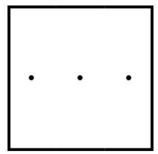
$\psi \in \{0,1\}$ is represented by the binary symbol image f_ψ,

$$f_\psi(x,y) = \begin{cases} 1, & \text{if } x,y = 0.5, \psi = 1 \\ 0, & \text{otherwise} \end{cases}$$



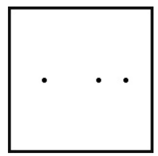
$w = w_1 w_2 \cdots w_k \in \Sigma^+$ is represented by the binary list image f_w,

$$f_w(x,y) = \begin{cases} 1, & \text{if } x = \frac{2i-1}{2k}, y = 0.5, w_i = 1 \\ 0, & \text{otherwise} \end{cases}$$



$w = w_1 w_2 \cdots w_k \in \{0,1\}^+$ is represented by the binary stack image f_w,

$$f_w(x,y) = \begin{cases} 1, & \text{if } x = 1 - \frac{3}{2^{k-i+2}}, y = 0.5, w_i = 1 \\ 0, & \text{otherwise} \end{cases}$$

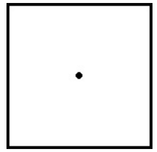


List/stack image f_w is said to have length $k \in \mathbb{N}$. (f_w, k) uniquely represents w.

Representing data as images

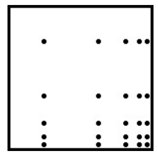
$r \in \mathbb{R}$ is represented by the real number image f_r

$$f_r(x,y) = \begin{cases} r, & \text{if } x,y = 0.5 \\ 0, & \text{otherwise} \end{cases}$$



$R \times C$ matrix A, with real-valued components a_{ij}, is represented by the $R \times C$ matrix image f_A

$$f_A(x,y) = \begin{cases} a_{ij}, & \text{if } x = 1 - \frac{1+2k}{2^{j+k}}, y = \frac{1+2l}{2^{i+l}} \\ 0, & \text{otherwise} \end{cases}$$



Language deciding by CSM

CSM M_L decides $L \subseteq \Sigma^*$ if M_L has initial configuration $\langle c_\mathrm{s}, g_\mathrm{s} \rangle$ and final configuration $\langle c_\mathrm{h}, g_\mathrm{h} \rangle$, and the following hold:

- sequence g_s contains the two input elements $(f_w, \iota_{1_\xi}, \iota_{1_\eta})$ and $(f_{1^{|w|}}, \iota_{2_\xi}, \iota_{2_\eta})$
- g_h contains the output element $(f_1, o_{1_\xi}, o_{1_\eta})$ if $w \in L$
- g_h contains the output element $(f_0, o_{1_\xi}, o_{1_\eta})$ if $w \notin L$
- $\langle c_\mathrm{s}, g_\mathrm{s} \rangle \vdash_M^* \langle c_\mathrm{h}, g_\mathrm{h} \rangle$, for all $w \in \Sigma^+$.

Where f_w is the binary stack image representation of $w \in \Sigma^+$, $f_{1^{|w|}}$ is the unary stack image representation of the unary word $1^{|w|}$. Images f_0 and f_1 are the binary symbol image representations of the symbols 0 and 1, respectively.

Analog recurrent neural networks

- Finite size feedback first-order neural networks with real weights

- Model of analog computation, by Siegelmann and Sontag, TCS, 1994

$$x_i(t+1) = \sigma\left(\sum_{j=1}^{N} a_{ij}x_j(t) + \sum_{j=1}^{M} b_{ij}u_j(t) + c_i\right) \quad , \quad i = 1, \ldots, N$$

$$\sigma(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \leq x \leq 1 \\ 1, & \text{if } x > 1 \end{cases} .$$

CSM simulation of ARNN

		sta					$\overline{u}$	$\Sigma\overline{AX}$	$\Sigma\overline{BU}$		t_1	a	b	t_2		O	I		**0**	**1**
	99	br	0	14																
(i)	14	ld	I	st	t_1a	st	I	ld	t_1	st	$\overline{u}$									
(ii)	13	ld	$\overline{x}$	whl	$\overline{N-1}$	st	t_3	ld	$\overline{x}$	ld	at_3	end								
(iii)	12	st	b	ld	$\overline{A}$	$\cdot$														
(iv)	11	st	t_2	ld	**0**	whl	$\overline{N-1}$	st	t_1	ld	t_2	st	bt_2	ld	t_1	$+$	end			
	10		st	b	ld	t_2	$+$	st	$\Sigma\overline{AX}$											
(v)	9	ld	$\overline{u}$	whl	$\overline{N-1}$	st	t_3	ld	$\overline{u}$	ld	at_3	end								
(vi)	8	st	b	ld	$\overline{B}$	$\cdot$														
(vii)	7	st	t_2	ld	0	whl	$\overline{M-1}$	st	t_1	ld	t_2	st	bt_2	ld	t_1	$+$	end			
	6		st	b	ld	t_2	$+$	st	$\Sigma\overline{BU}$											
(viii)	5	ld	$\Sigma\overline{AX}$	st	b	ld	$\Sigma\overline{BU}$	$+$	st	b	ld	$\overline{c}$	$+$							
(ix)	4	ρ	$\widehat{\mathbf{0}}$	$\widehat{\mathbf{1}}$	st	t_3	ld	**0**	st	t_1										
(x)	3	whl	$\overline{N-1}$	ld	t_3	st	at_3	ld	t_1a	st	t_1	end	ld	t_3	st	ab	ld	t_1a	st	t_1
	2	whl	$\overline{N-1}$	ld	t_1	st	t_1a	ld	ab	st	b	end	ld	t_1	st	t_1a	ld	ab	st	$\overline{x}$
(xi)	1	st	b	ld	$\overline{P}$	$\cdot$	st	t_1	ld	O	ld	t_1a	st	O						
(xii)	0	br	0	14																
		0	1	2	3	4	5	6	7	8	9	10	...	$\overline{x}$	$\overline{N-1}$	$\overline{M-1}$	$\overline{A}$	$\overline{B}$	$\overline{c}$	$\overline{P}$

note: address t_3 is located at grid coordinates $(10, 14)$

CSM simulation of ARNN

$$
\begin{array}{llcl}
\text{(i)} & \overline{u} & := & I.\text{pop}() \\
\text{(ii)} & \overline{X} & := & \text{push } \overline{x} \text{ onto itself vertically } N-1 \text{ times} \\
\text{(iii)} & \overline{AX} & := & \overline{A} \cdot \overline{X} \\
\text{(iv)} & \Sigma\overline{AX} & := & \Sigma_{i=1}^{N} \left(\overline{AX}.\text{pop}_i()\right) \\
\text{(v)} & \overline{U} & := & \text{push } \overline{u} \text{ onto itself vertically } N-1 \text{ times} \\
\text{(vi)} & \overline{BU} & := & \overline{B} \cdot \overline{U} \\
\text{(vii)} & \Sigma\overline{BU} & := & \Sigma_{i=1}^{M} \left(\overline{BU}.\text{pop}_i()\right) \\
\text{(viii)} & \text{affine-comb} & := & \Sigma\overline{AX} + \Sigma\overline{BU} + \overline{c} \\
\text{(ix)} & \overline{x}' & := & \rho(\text{affine-comb}, \mathbf{0}, \mathbf{1}) \\
\text{(x)} & \overline{x} & := & (\overline{x}')^T \\
\text{(xi)} & & & O.\text{push}\,(\overline{P} \cdot \overline{x}) \\
\text{(xii)} & & & \text{goto step (i)}
\end{array}
$$

CSM decides any $L \subseteq \{0,1\}^+$

- Formal nets; a class of ARNNs that decide languages
- For each $L \subseteq \{0,1\}^+$ there exists formal net $\mathcal{F}_L$ that decides L
- We carry this result over to the CSM by giving a CSM $\mathcal{D}$ that
  - is consistent with the definition of language deciding by CSM
  - decides L by simulating $\mathcal{F}_L$

CSM $\mathcal{D}$

	sta					$\overline{u}$	$\Sigma\overline{AX}$	$\Sigma\overline{BU}$		t_1	a	b	t_2		f_{ψ_w}	f_w	$f_{1\|w\|}$	**0**	**1**
99	br	0	18																
18	ld	f_w	st	b	ld	**0**	st	t_2											
17	whl	$f_{1\|w}$	ld	b	st	t_1a	st	b	ld	t_2	ld	t_1a	st	t_2	end	st	f_w		
16	ld	f_w	st	t_1a	st	f_w	ld	t_1	st	13	16	14	14	ld	14	17	14	14	
15	st	b	ld	$f_{1\|w}$	st	t_1a	st	$f_{1\|w\|}$	ld	t_1	st	13	16	14	14				
14	ld	12	15	14	14	$+$	st	$\overline{u}$	br	0	13								
⋮																			
1	st	b	ld	$\overline{P}$	$\cdot$	st	t_3	st	t_1	whl	$\overline{O_v}$	st	t_1a	end	ld	t_1	br	0	$\widehat{\text{a}}$
f_1	ld	t_3	whl	$\overline{O_d}$	st	t_1a	end	ld	t_1	st	f_{ψ_w}	hlt							
f_0	br	0	16																
	0	1	2	3	4	5	6	7	8	...	$\overline{O_d}$	$\overline{O_v}$	$\overline{x}$	$\overline{N-1}$	$\overline{M-1}$	$\overline{A}$	$\overline{B}$	$\overline{c}$	$\overline{P}$

note: address t_3 is located at grid coordinates $(10, 18)$

CSM $\mathcal{D}$

CSM $\mathcal{D}$, in the worst case, requires

- linear TIME

$$T(N, M, \boldsymbol{T}(|w|), |w|, d, v) = 12|w| + 7d + (49N + 7v + 67)\boldsymbol{T}(|w|) + 22$$

- exponential RESOLUTION

$$R(N, M, \boldsymbol{T}(|w|), |w|, d, v) = \max(2^{|w|}, 2^{(2N-2)})$$

- constant SPACE

- infinite RANGE ω

Needle in haystack problem

- Given $w \in 0^*10^*$, what is the index of the '1' in w?
- Conventional (serial) computer requires $\Theta(n)$ steps, worst case
- Grover's quantum computer algorithm requires $O(\sqrt{n})$ comparisons, average case
- CSM algorithm requires $\Theta(\log_2 n)$ steps, worst case

Needle in haystack problem

	i1	i2			f_0	f_1			sta					a	b	c	d	e									
99									br	0	3																
w	ld	e	st	de	br	0	*d																				
3	ld	i2	st	e	ld	f_0	st	c	br	0	w																
2	h	v	st	b	*	·	h	v	br	8	*a																
1	ld	i1	st	ab	br	0	2		ld	i1	st	ab	st	i1	ld	f_0	st	b	ld	bc	st	c	br	0	w		
0	ld	c	hlt						ld	i1	st	ab	ld	b	st	i1	ld	f_1	st	b	ld	bc	st	c	br	0	w
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...									

Needle in haystack problem

```
(8,99)    procedure search(i1, i2)
(0,3)       e := i2
(4,3)       c := $f_0$
(0,w)       while (e.pop() = $f_1$)
(0,1)         rescale i1 over both image a and image b
(0,2)         FT, square, and FT image a
(8,2)         if (a = $f_1$)
(8,1)           i1 := LHS of i1
(14,1)          c.push($f_0$)
(8,2)         else /* a = $f_0$ */
(8,0)           i1 := RHS of i1
(16,0)          c.push($f_1$)
              end if
            end while
(0,0)       a := c
          end procedure
```

Needle in haystack problem

On input word of length n, CSM needle in haystack algorithm, in the worst case, requires

- log TIME, $T(n) = 23 \log_2 n + 11$
- linear RESOLUTION, $R(n) = 2n$
- constant SPACE
- constant RANGE

Future work

- Prove further computability and complexity results
- Investigate (computationally less powerful) variants of the CSM

Summary

- Presented the continuous space machine
- Analog recurrent neural network simulation
- A log time solution to the needle in haystack problem
- Acknowledgements: TASS, IRCSET

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